INTRODUCTION TO TROPICAL GEOMETRY

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INTRODUCTION

The aim of these notes is to introduce the subject of tropical geometry and subsequently describe the correspondence between algebraic varieties and tropical varieties.

There is only one section and it is devoted to introduce the tropical linear algebra.

1. TROPICAL LINEAR ALGEBRA

A *tropical vector space* V (T-vector space) is a set endowed with two operations, $\oplus_V : V \times V \to V$ and $\odot_V : \mathbb{R}_{\min} \times V \to V$ with the following properties:

(1) V is a monoid with respect to $\oplus_V$,
(2) $a \odot_V v \in V$ for any $a \in \mathbb{R}_{\min}$ and $v \in V$,
(3) $(a \odot b) \odot_V v = a \odot_V (b \odot_V v)$ for any $v \in V$,
(4) $(a \oplus b) \odot_V v = (a \odot_V v) \oplus_V (b \odot_V v)$ for any $a, b \in \mathbb{R}_{\min}$ and $v \in V$,
(5) $a \odot_V (v_1 \oplus_V v_2) = (a \odot_V v_1) \oplus_V (a \odot_V v_2)$ for any $a \in \mathbb{R}_{\min}$ and $v_1, v_2 \in V$,
(6) $0 \odot_V v = v$ for any $v \in V$.

In what follows we will use the notation $\oplus, \odot$ for $\oplus_V, \odot_V$. The elements of the tropical vector space will be called tropical vectors or T-vectors.

1.1. *Exercise.* — Use the preceding axioms for proving that $+\infty \odot v = u$, for any $v \in V$, where u is the additive unity of the T-vector space V.

1.2. *Example.* — The set of n-tuples of elements of $\mathbb{R}_{\min}$ can be given a T-vector space structure by adding, resp. multiplying, component per component. This T-vector space will be denoted by $\mathbb{R}^n_{\min}$.

A *tropical basis* for a T-vector space is a minimal set of vectors such that any element of V can be written as a tropical linear combination of its elements.

1.3. *Proposition.* — Any base of $\mathbb{R}^n_{\min}$ has the form: $e_1, \ldots, e_n$ where e_i has all but the i-coordinate equal to $+\infty$.

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Proof. Let b_i be the i-coordinate of e_i, then any element $(a_1, \dots, a_n)$ of $\mathbb{R}^n_{\min}$ can be written as $(a_1 - b_1) \odot e_1 \oplus \cdots \oplus (a_n - b_n) \odot e_n$.
On the other hand, if $f_1, \dots, f_n \in \mathbb{R}_{\min}$ is another base then from the equality

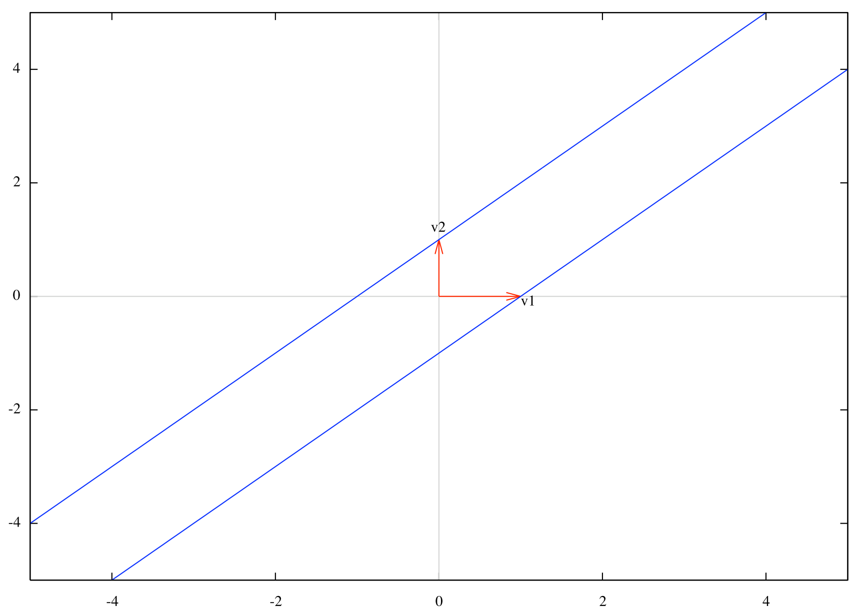
$$e_1 = a_{11} \odot f_1 \oplus \cdots \oplus a_{1n} \odot f_n,$$

we deduce that the coefficient of f_i must be $+\infty$ if f_i is not a multiple of e_1. This means that the preceding equality reduces to $e_1 = a_{1i} \odot f_i$. $\square$

1.4. *Example.* — Let V be the T-vector space generated by the two vectors $v_1 = (1, 0)$ and $v_2 = (0, 1)$. In order to visualize this space, observe that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + h \\ y \end{pmatrix} \oplus \begin{pmatrix} x \\ y + k \end{pmatrix}$$

where $h, k \geq 0$ and moreover that there are no other ways of decomposing $v = (x, y)^T$ as a T-sum of two vectors distinct from v itself. This implies that the equation $v = a \odot v_1 \oplus b \odot v_2$ has a solution only if $a \odot v_1$ and $b \odot v_2$ belongs to the horizontal and vertical positive half lines departing from v. In other words v has to be a point inside the strip bounded by the two lines $v_1 + (a, a)^T$, $v_2 + (b, b)^T$.



1.5. *Question.* — Suppose that a T-vector space V admits a basis of cardinality r, is it true that all its bases have the same cardinality?

1.6. *Example.* — Let $v_1 = (1, 0)^T, v_2 = (0, 1)^T$ and $e_1 = (+\infty, 0)^T, e_2 = (0, +\infty)^T$, then

$$\langle v_1, v_2 \rangle \subset \langle e_1, e_2 \rangle$$

and both the spaces have dimension 2.

Before going on we need the analogue of another classical notion from linear algebra.

A *tropical linear map* $f : V \to W$ is a map such that

$$f(a \odot v_1 \oplus b \odot v_2) = a \odot f(v_1) \oplus b \odot f(v_2)$$

1.7. *Example.* — A T-linear map $f : \mathbb{R}_{\min} \to \mathbb{R}_{\min}$ must satisfy

$$f(x) = f(x \odot 0) = x \odot f(0) = x + f(0)$$

hence, if $f(0) \neq +\infty$, the map is just a translation.

Two T-vector spaces will be said *isomorphic* if there exists an invertible T-linear map between them.

As for the image of a T-linear map we have the following.

1.8. *Proposition.* — The image of a T-linear map is a T-vector space.

Observe that there is no natural notion of the kernel of a T-map, since the equation $f(v) = +\infty$ has $v = +\infty$ as its only solution.

The classical correspondence between linear maps and matrices has an analogue in the tropical context. First of all let us start with the basic definitions and operations between matrices.

Let $A = (a_{ij})$ be an $m \times n$ matrix and let $B = (b_{ij})$ be an $n \times r$ one. We define the *tropical product of A and B* in the usual way:

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \odot \begin{pmatrix} b_{11} & \dots & b_{1r} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nr} \end{pmatrix} = \begin{pmatrix} c_{11} & \dots & c_{1r} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mr} \end{pmatrix}$$

where

$$c_{ij} = a_{i1} \odot b_{1j} \oplus \cdots \oplus a_{in} \odot b_{nj}.$$

1.9. *Example.* — Let $a, b \geq 1$ and consider the product

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This means that the left hand side matrix induces the identity on the T-vector space generated by the columns of the right hand side.

Of course there are also automorphisms which are non-trivial, for example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In this case the left hand side matrix switch the two generators of the T-vector space.

An easy consequence of the matrix-product definition is the following:

1.10. *Proposition.* — Let A be a $n \times m$ matrix, and let $V \subseteq \mathbb{R}^m_{\min}$ be a T-vector space, then the map $v \to Av$, for any $v \in V$, is a T-linear map.

To any matrix A we can associated a T-linear map $f : V_1 \to V_2$ and to this map we can associate a T-vector space: $f(V_1)$. A natural question at this point would be to ask about the dimension of $f(V_1)$. As a first step in this direction we can consider thee case in which A is a square matrix and define its *T-determinant* to be:

$$\mathrm{Det}(A) = \bigoplus_{\sigma \in S_n} a_{1\sigma(1)} \odot \cdots \odot a_{1\sigma(n)}$$

where σ varies over all the permutations S_n of a set of n elements.

Observe that the T-determinant is a minimum of a set of linear forms in the matrix entries. A matrix M is said to be *T-singular* if the minimum of the linear forms used for evaluating its determinant is attained at least twice.

1.11. *Example.* — The determinants of the three matrices

$$A = \begin{pmatrix} 0 & 5 \\ 0 & 6 \end{pmatrix} \quad B = \begin{pmatrix} +\infty & 5 \\ +\infty & 6 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

are $5, +\infty$ and 2 respectively, and both the C and D are singular.

1.12. *Remark.* — Observe that if the a matrix A has determinant $+\infty$ then it is singular.

The relation between the T-determinant of a matrix and the image of a T-linear map associated to A is given by the following:

Given a tropical matrix A we define its *kernel* to be the set of those $n \times 1$ matrices $v = (v_1, \dots, v_n)^T$, i.e. *tropical vectors*, such that the minimum

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \odot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \min\{a_{11} + v_1, \dots, a_{1n} + v_n\} \\ \vdots \\ \min\{a_{m1} + v_1, \dots, a_{mn} + v_n\} \end{pmatrix}$$

is attained at least twice.

1.13. **Exercises.**

(1) Given a 2×2 square matrix A find all the matrices M such that $MA = A$.

(2) Prove or disprove the following statement: the tropical product AB of 2×2 matrices is singular if and only if either A or B are singular.

(3) Classify all the isomorphism equivalence classes of T-vector spaces $V \subseteq \mathbb{R}^2_{\min}$.

(4) Let A be a singular 2×2 matrix. What can we say about its T-image? And what about its T-kernel?

(5) Determine the T-image and the T-kernel of the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

References

[1] Develin, M. and Sturmfels, B. *Tropical convexity.* math.AG/0308254
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