Curves Seminar - Fall 2005
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Lecture 2

1. Cremona transformations of $\mathbb{P}^2$

In this part we will consider *rational* transformations of $\mathbb{P}^2$. A rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is defined by means of three homogeneous polynomials of degree d, $P_i(x_0, x_1, x_2)$ for $i = 0, \dots, 2$ in the following way:

$$\phi([x_0 : x_1 : x_2]) := [P_0(x_0, x_1, x_2) : P_1(x_0, x_1, x_2) : P_2(x_0, x_1, x_2)].$$

Observe that ϕ is well defined outside the set $\{x \in \mathbb{P}^2 \mid P_0(x) = P_1(x) = P_2(x) = 0\}$. This set is called the *indeterminacy locus* of ϕ. Any rational map which can be inverted is called *birational*.

Remark 1.1. If the indeterminacy locus of ϕ has dimension 1, i.e. it contains a curve, then the three polynomials P_i's contain a common factor. In this case we will substitute each $P_i(x)$ with $P_i(x)' := P_i(x)/Q(x)$ where $Q(x) = \gcd\,(P_1(x), P_2(x), P_3(x))$. The map ϕ' associated to the P_i''s will have an indeterminacy locus of dimension at most zero.

Consider the rational map

$$\sigma([x_0 : x_1 : x_2]) := [x_1x_2 : x_0x_2 : x_0x_1]. \qquad (1)$$

Since $\sigma^2 = 1_{\mathbb{P}^2}$ it follows that σ is birational.

Remark 1.2. The indeterminacy locus of σ is given by the three points

$$p_0 = [1 : 0 : 0],\ p_1 = [0 : 1 : 0],\ p_2 = [0 : 0 : 1].$$

Moreover, the three lines given of equation $x_i = 0$, for $i = 0, \dots, 2$, are contracted to three points. Observe that the polynomials defining σ can be considered as a base of the polynomials of the degree 2 piece of the homogeneous ideal $I_{p_0} \cap I_{p_1} \cap I_{p_2}$. This means that σ is the transformation associated to the linear system $\mathcal{L}_2(2; 1^3)$ through the p_i's.

A *quadratic transformation* is a rational transformation associated to a linear system $\mathcal{L}_2(2; 1^3)$ through three non-collinear points. Such a transformation is said to be *centered* at the p_i's which are the indeterminacy locus of the transformation.

The following theorem will clarify why these maps are so important:

Theorem 1.3. *(Noether) Any birational map ϕ of $\mathbb{P}^2$ can be written as*

$$\phi = g \circ \sigma_n \circ \cdots \circ \sigma_1$$

where the σ_i's are quadratic transformations centered at different points and $g \in \mathrm{PGL}(3, \mathbb{C})$.

2. The action of σ on linear systems

Let $\mathcal{L} = \mathcal{L}_2(d; m_1, \dots, m_r)$ be a linear system through fat points. We want to study the action of a quadratic transformation on $\mathcal{L}$.

Proposition 2.1. *Let $\mathcal{L} = \mathcal{L}_2(d; m_1, \dots, m_r)$ be a linear system through multiple points in general position and let $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a quadratic transformation centered at the first three multiple points of the system, then*

$$\sigma^*\mathcal{L} \;=\; \mathcal{L}_2(d+k; m_1+k, \dots, m_3+k, m_4, \dots, m_r),$$

where $k = d - m_1 - m_2 - m_3$.

Proof. By a linear change of coordinates, we can suppose that the first 3 multiple points are $[1:0:0], [0:1:0], [0:0:1]$ so that σ is expressed by (1). In order to simplify the further notation we redefine these three multiplicities as m_0, m_1, m_2. For each polynomial f defining a curve of the system $\mathcal{L}$ one has that

$$f(x_0, x_1, x_2) = \sum_{a_0+a_1+a_2=d} x_0^{a_0} x_1^{a_1} x_2^{a_2}.$$

Moreover, in order to have multiplicities m_0, m_1, m_2 at the three fundamental points, each monomial $x_0^{a_0} x_1^{a^1} x_2^{a_2}$ of f has to satisfy the following:

$$a_0 + a_1 \geq m_2, \qquad a_0 + a_2 \geq m_1, \qquad a_1 + a_2 \geq m_0,$$

or equivalently

$$a_0 \leq d - m_0, \qquad a_1 \leq d - m_1, \qquad a_2 \leq d - m_2.$$

Observe that the set $T \subset \mathbb{N}^3$ of points (a_0, a_1, a_2) satisfying these inequalities is contained in a triangle.

The action of σ on each monomial is described by

$$\sigma^*(\prod_{i=0}^{2} {x_i}^{a_i}) = \prod_{i=0}^{2} {x_i}^{d-a_i} = (x_0^{m_0} x_1^{m_1} x_2^{m_2}) \prod_{i=0}^{2} {x_i}^{d-a_i-m_i}.$$

This may be summarized by saying that σ^* induces the map

$$f(a_0, a_1, a_2) = (d - a_0 - m_0, d - a_1 - m_1, d - a_2 - m_2)$$

from $\mathbb{N}^3$ to $\mathbb{N}^3$. This map is a bijection and its image can be described as the set of $(a_0', a_1', a_2') \in \mathbb{N}^3$ such that $a_0' + a_1' + a_2' = d'$ and

$$a_0' \leq d' - m_0', \qquad a_1' \leq d' - m_1', \qquad a_2' \leq d' - m_2'.$$

where $d' = d + k$ and $m_i' = m_i + k$. □

Remark 2.2. Observe that if we apply a quadratic transformation to the first three points of $\mathcal{L}_2(d; m_1, \dots, m_r)$ and some of the numbers $d - m_i - m_j$ $(1 \leq i < j \leq 3)$ are negative then we obtain a negative multiplicity in the transformed system. In this case the line $\langle p_i, p_j \rangle$ is a fixed line of the linear system and it is contained $a_{ij} = m_i + m_j - d$ times in the base locus of $\mathcal{L}$. Observe that the image of the multiple line $\mathcal{L}_2(a; a^2)$ under a quadratic transformation based at the two points is $\mathcal{L}_2(0; -a)$.

The preceding remark suggests to change the notation adopted for defining linear systems by allowing also negative multiplicities. Each of these negative multiplicity will represent the quadratic transform of a fixed line contained into the system.
The following shows that quadratic transformations behave well with respect to the virtual and effective dimension of linear systems.

Proposition 2.3. *Given a linear system $\mathcal{L} = \mathcal{L}_2(d; m_1, \ldots, m_r)$ and a quadratic transformation σ we have $(\sigma^*\mathcal{L})^2 = \mathcal{L}^2$, moreover*

$$\dim \sigma^*\mathcal{L} = \dim \mathcal{L} \qquad \text{and} \qquad v(\sigma^*\mathcal{L}) = v(\mathcal{L}).$$

Proof. For proving the first part, it is enough to consider a quadratic transformation σ centered at the first three points of $\mathcal{L}$. In this case one has

$$\begin{aligned}(\sigma^*\mathcal{L})^2 - \mathcal{L}^2 &= (d+k)^2 - \sum_{i=1}^{3}(m_i+k)^2 - (d^2 - \sum_{i=1}^{3} m_i^2) \\ &= 2dk + k^2 - 2k\sum_{i=1}^{3} m_i - 3k^2 \\ &= 2k(d - \sum_{i=1}^{3} m_i - k) \\ &= 0.\end{aligned}$$

Observe that $\dim \mathcal{L}$ is preserved by any birational transformation ϕ, since the curves of $\mathcal{L}$ are in one to one correspondence with those in $\phi^*(\mathcal{L})$.
For the virtual dimension, since $v(\mathcal{L}) = (\mathcal{L}^2 - K\mathcal{L})/2$, it is enough to prove that $\sigma^*(-K) = -K$, but this is an easy consequence of Proposition 2.1. □

The preceding proposition implies that σ^* is an isometry of the quadratic form associated to the intersection form.

Remark 2.4. Observe that $-K$ and its multiples are the only linear systems for which $\sigma^*(\mathcal{L}) = \mathcal{L}$ for any σ based at any three points of the system.

Example 2.5. In the preceding lecture we found two special linear systems through double points: $\mathcal{L}_2(2; 2^2)$ and $\mathcal{L}_2(4; 2^5)$. By applying a quadratic transformation to the last system based at any three double points, one obtains the first system.

Example 2.6. The system $\mathcal{L}_2(12; 5^6)$ has virtual dimension 0 and $k = 12 - 15 = -3$. Hence, by applying a quadratic transformation to any three points one obtains the system $\mathcal{L}_2(9; 5^3, 2^3)$. Observe that this new system has negative intersection with the three lines through the three points of multiplicity 5, so these lines are contained in the base locus of the system. Performing a quadratic transformation based on the three points of multiplicity 5 one obtains the system $\mathcal{L}_2(3; 2^3, -1^3)$. By applying another quadratic transformation to the double points the system transforms into $\mathcal{L}_2(0; -1^6)$. This means that the original system is made by six fixed components which are transformations of a line through two points (so that they are (-1)-curves) and that each component has intersection -1 with the system. Actually these components are the six conics through 5 of the six multiple points.

Looking at the preceding example one can see that any two fixed conics of the system have intersection 0 since they share only 4 points. It turns out that this is always the case for any two (-1)-curves contained in the base locus of a linear system.

Proposition 2.7. *Let C_1, C_2 be two distinct (-1)-curves contained in the base locus of a linear system $\mathcal{L}$, then $C_1C_2 = 0$.*

Proof. Observe that the linear system $|C_1 + C_2|$ has dimension 0 since it is contained in the fixed part of $\mathcal{L}$ and this implies that $v(C_1 + C_2) \leq 0$. By means of the formula

$$v(C_1 + C_2) = v(C_1) + v(C_2) + C_1C_2,$$

one obtains that $C_1C_2 \leq 0$, since by definition of (-1)-curve one has that $v(C_i) = 0$ for $i = 1, 2$. On the other hand, since C_1 and C_2 are irreducible and distinct then, by Bezout theorem, they must intersect in finitely many points and this implies that $C_1C_2 \geq 0$. □

Example 2.8. Consider the system $\mathcal{L}_2(9; 5, 4, 3^5)$ which has virtual dimension -1. After applying a quadratic transformation based on the first three points one obtains the system $\mathcal{L}_2(6; 0, 1, 2, 3^4)$ which can be written as $\mathcal{L}_2(6; 3^4, 2, 1)$ after sorting the multiplicities of the points. This last system can be still transformed into $\mathcal{L}_2(3; 3, 2, 1)$. By applying one more quadratic transformation one obtains $\mathcal{L}_2(0; -1, -2)$. In this way we can see that our original system was composed of two (-1)-curves, one contained 1-time in the system and the other contained 2-times.
In order to recognize these curves in the starting system one has to make all the quadratic transformations in the reverse order:

$$\begin{array}{lrrrrrrr}
\mathcal{L}_2(0; & \boxed{-1}, & \boxed{-2}, & 0, & 0, & 0, & 0, & \boxed{0}) \\
\mathcal{L}_2(3; & 2, & 1, & 0, & \boxed{0}, & \boxed{0}, & \boxed{0}, & 3) \\
\mathcal{L}_2(6; & \boxed{2}, & \boxed{1}, & \boxed{0}, & 3, & 3, & 3, & 3) \\
\mathcal{L}_2(9; & 5, & 4, & 3, & 3, & 3, & 3, & 3)
\end{array}$$

which gives:

$$\begin{array}{lrrrrrrrclrrrrrrr}
\mathcal{L}_2(0; & \boxed{-1}, & \boxed{0}, & 0, & 0, & 0, & 0, & \boxed{0}) & + & 2\mathcal{L}_2(0; & \boxed{0}, & \boxed{-1}, & 0, & 0, & 0, & 0, & \boxed{0}) \\
\mathcal{L}_2(1; & 0, & 1, & 0, & \boxed{0}, & \boxed{0}, & \boxed{0}, & 1) & + & 2\mathcal{L}_2(1; & 1, & 0, & 0, & \boxed{0}, & \boxed{0}, & \boxed{0}, & 1) \\
\mathcal{L}_2(2; & \boxed{0}, & \boxed{1}, & \boxed{0}, & 1, & 1, & 1, & 1) & + & 2\mathcal{L}_2(2; & \boxed{1}, & \boxed{0}, & \boxed{0}, & 1, & 1, & 1, & 1) \\
\mathcal{L}_2(3; & 1, & 2, & 1, & 1, & 1, & 1, & 1) & + & 2\mathcal{L}_2(3; & 2, & 1, & 1, & 1, & 1, & 1, & 1)
\end{array}$$

3. (−1)-CURVES

The aim of this section is to prove that all the (−1)-curves are in the same orbit with respect to the action of the group generated by quadratic transformations.

Proposition 3.1. *For any* (-1)*-curve* E *either* $E \in \mathcal{L}_2(1;1^2)$ *or there exists a quadratic transformation* σ *such that* $\deg \sigma^*(E) < \deg E$*.*

Proof. Let $E \in \mathcal{L}_2(\delta;\mu_1,\dots,\mu_r)$ and assume that the multiplicities have been sorted in decreasing order, i.e. $\mu_1 \geq \cdots \geq \mu_r$. From the equations

$$\delta^2 - \sum_{i=1}^{r} \mu_i^2 = -1 \tag{2}$$

$$3\delta - \sum_{i=1}^{r} \mu_i = 1, \tag{3}$$

one can deduce that

$$\delta^2 - 3\mu_3\delta - \sum_{i=1}^{3}(\mu_i^2 - \mu_3\mu_i) = \sum_{4}^{r}(\mu_i^2 - \mu_3\mu_i) - 1 - \mu_3 < 0,$$

which means that

$$\delta(\delta - 3\mu_3) - \sum_{i=1}^{3}(\mu_i^2 - \mu_3\mu_i) < 0.$$

We want to prove that the preceding inequality implies that $\delta < \mu_1 + \mu_2 + \mu_3$. Assume the contrary, then $\delta \geq \mu_1 + \mu_2 + \mu_3$ and in particular $\delta \geq 3\mu_3$. Since E is effective then $\delta \geq \mu_1$ so, by substituting this value for δ in the preceding equation, one obtains:

$$\mu_1(\delta - 3\mu_3) - \sum_{i=1}^{3}(\mu_i^2 - \mu_3\mu_i) < 0$$

and by substituting $\mu_1 + \mu_2 + \mu_3$ to δ one has

$$\begin{aligned} & \mu_1(\mu_1 + \mu_2 + \mu_3 - 3\mu_3) - \mu_1^2 - \mu_2^2 + \mu_3\mu_1 + \mu_3\mu_2 \\ = \; & \mu_1\mu_2 - \mu_1\mu_3 - \mu_2^2 + \mu_2\mu_3 \\ = \; & (\mu_1 - \mu_2)(\mu_2 - \mu_3) < 0 \end{aligned}$$

which is a contradiction. □

The preceding proposition implies that a quadratic transformation σ can be applied to E in order to decrease its degree. Since σ^*E is still a (−1)-curve, reasoning as before, one can decrease its degree another time. The procedure goes on until the third biggest multiplicity is positive and it stops as soon as the last μ_3' is 0. Hence we are left with a (−1)-curve through two multiple points. In this case the equality $3\delta' - \mu_1' - \mu_2' = 1$ implies that $\delta' \leq \mu_1' + \mu_2' - 1$ and this means that the line through the two multiple points is a component of the system. From the fact that E is irreducible and reduced we deduce that this last system has to be $\mathcal{L}_2(1;1^2)$.

Corollary 3.2. *Given any* (-1)*-curve* E *there exists a sequence of quadratic transformations* $\sigma_1, \dots, \sigma_n$ *such that the sequence of* (-1)*-curves defined by:* $E_0 = E$ *and* $E_i = \sigma_i^*(E_{i-1})$ *for* $i = 1, \dots, n$ *has the following properties:* $\deg E_i < \deg E_{i-1}$ *and* $E_n \in \mathcal{L}_2(1; 1^2)$.

Before going on we need the following definition:

Definition 3.3. A linear system $\mathcal{L}_2(d; m_1, \dots, m_r)$ is in *standard form* if $m_1 \geq \cdots \geq m_r \geq 0$ and $d \geq m_1 + m_2 + m_3$.

The fact that the action of the group generated by quadratic transformations is transitive on the set of (-1)-curves has an important consequence on the study of linear systems.

Proposition 3.4. *Let* $\mathcal{L} = \mathcal{L}_2(d; m_1, \dots, m_r)$ *be a linear system in standard form, then* $\mathcal{L}E \geq 0$ *for any* (-1)*-curve* E.

Proof. It is enough to prove that if E is a (-1)-curve and σ is a quadratic transformation such that $\deg \sigma^*(E) < \deg E$ then $\mathcal{L}\sigma^*(E) \leq \mathcal{L}E$. In fact, by using the sequence constructed in Corollary 3.2, one would obtain that

$$\mathcal{L}E_n \leq \cdots \leq \mathcal{L}E_0 = \mathcal{L}E,$$

where $E_n \in \mathcal{L}_2(1; 1^2)$. Since the system is in standard form then $\mathcal{L}E_n = d - m_1 - m_2 \geq 0$.
The required inequality can be proved by a direct calculation:

$$\begin{aligned} \mathcal{L}E - \mathcal{L}\sigma^*(E) &= d\delta - \sum_{i=1}^{3} m_i\mu_i - (d(\delta + k) - \sum_{i=1}^{3} m_i(\mu_i + k)) \\ &= -k(d - \sum_{i=1}^{3} m_i) \end{aligned}$$

is ≥ 0 since $k = \delta - \mu_1 - \mu_2 - \mu_3 < 0$ and $\mathcal{L}$ is in standard form. □

4. An effective conjecture

In this section we state a conjecture about special linear systems on the blowing-up of $\mathbb{P}^2$ at general points. This conjecture allows one to easily decide if a linear system is special. We will prove that this conjecture is equivalent to the Gimigliano-Harbourne-Hirschowitz one.

Conjecture 4.1. *A linear system* $\mathcal{L}_2(d; m_1, \dots, m_r)$ *in standard form is non-special.*

It is easy to see that if Conjecture 4.1 is true than there exists an algorithm for deciding if a linear system is special or not. Moreover one has the following.

Theorem 4.2. *Conjecture 4.1 is equivalent to the G.H.H. conjecture.*

Proof. Suppose that $\mathcal{L} = \mathcal{L}_2(d; m_1, \dots, m_r)$ is in standard form, then by 3.4 it can not have negative intersection with any (-1)-curve, hence by G.H.H. it is non-special.
On the other hand if a linear system $\mathcal{L}$ is special then by Conjecture 4.1 it can not be in standard

form. This implies that, after sorting the multiplicities in decreasing order, it is possible to apply a quadratic transformation to the first three points of $\mathcal{L}$ which decrease its degree. After a finite number of these steps one obtains a new linear system of the form $\mathcal{L}_2(d'; m'_1, \ldots, m'_r)$ which may contain negative multiplicities and to which is no longer possible to apply a quadratic transformation in order to decrease its degree. If $\max(m'_1, \ldots, m'_r) \geq -1$ then each (-1)-curve has intersection at most -1 with $\mathcal{L}$ and after removing these curves from $\mathcal{L}$, the residual linear system has the same virtual dimension of $\mathcal{L}$. Moreover this system is in standard form and it can not be special, which gives a contradiction. So for at least one i, $m'_i \leq -2$ and this means that the corresponding (-1)-curve E_i gives $\mathcal{L}E_i \leq -2$. □