INTRODUCTION TO TROPICAL GEOMETRY

ANTONIO LAFACE

Introduction

The aim of these notes is to introduce the subject of tropical geometry and subsequently describe the correspondence between algebraic varieties and tropical varieties.

In section 1 we recall the correspondence between algebraic varieties defined over a non-archimedean field and tropical varieties. The aim of section 2 is to describe the local and global behavior of a tropical plane curve. Finally section 3 is devoted to the study of tropical conics and cubics.

1. From classical to tropical geometry

The content of the past lecture was devoted to establish a connection between classical and tropical geometry. Let us review in which way this can be done.

First of all we need to work over an algebraically closed field K of characteristic 0 which is endowed with a valuation $v(z) = -\log|z|$, where $|\ |$ is a non-archimedean norm. Let us consider the map $\mathrm{val} : (K^*)^n \to \mathbb{R}^n$ defined as:

$$\mathrm{val}(z_1, \dots, z_n) = (-\log|z_1|, \dots, -\log|z_n|),$$

and let $X \subset (K^*)^n$ be an algebraic variety, we define the *tropical variety* associated to X to be:

$$T(X) := \mathrm{val}(X).$$

The val map allows us to construct a dictionary between the two classes of varieties:

$$\left\{ \begin{array}{c} \text{Algebraic varieties} \\ X \subset (K^*)^n \end{array} \right\} \xrightarrow{\mathrm{val}} \left\{ \begin{array}{c} \text{Piecewise linear varieties} \\ T(X) \subset \mathbb{R}^n \end{array} \right\}$$

1.1. *Remark.* — The reason for this definition relies on the fact that if X is the zero locus of a Laurent polynomial $\sum a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$, then $T(X)$ is the tropical hypersurface (the only type of tropical variety that we have defined up to now) associated to $\bigoplus_{\mathbf{i}} v(a_{\mathbf{i}}) \odot \mathbf{x}^{\mathbf{i}}$:

$$\{\mathbf{z} \in (K^*)^n \mid \sum a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}} = 0\} \xrightarrow{\mathrm{val}} \left\{ \begin{array}{c} \mathbf{x} \in \mathbb{R}^n \mid \min_{\mathbf{i}}\{v(a_{\mathbf{i}}) + \mathbf{x} \cdot \mathbf{i}\} \\ \text{is attained at least twice} \end{array} \right\}$$

This correspondence can be explored in many directions. Here we will concentrate our attention on the case of algebraic curves in $(K^*)^2$. The embedding $(K^*)^2 \subset \mathbb{P}^2_K$ provides a natural compactification for algebraic curves $C \in (K^*)^2$. In this way we are allowed to speak the *genus* and the *degree* of C as being those which come from the compactified

Date: October 18, 2006.

$\bar{C} \subset \mathbb{P}^2_K$. It is possible to give a notion of *T-degree* and *T-genus* for T-curves. It will be the case that the correspondence between curves and T-curves will preserve these invariants.

1.2. *Remark.* — The connected component of the group $\mathrm{Aut}((K^*)^n)$ acts naturally on algebraic subvarieties of $(K^*)^n$. This action, through the val map, corresponds to the group of translations of $\mathbb{R}^n$, so that it is naturally to consider two T-varieties to be *linearly isomorphic* if and only if one can be obtained from the other by means of a translation.

1.3. **Exercises.**

(1) Describe the action of the image of $\mathrm{Aut}((K^*)^2)$ through the val map on tropical plane curves.

2. Tropical plane curves

In what follows, we will work over the $\max$ tropical semiring, which is the one where

$$x \oplus y = \max\{x, y\}.$$

The reason for this change will be evident as soon as we will start to describe the local behavior of a tropical algebraic curve. In order to obtain the $\min$ version of the tropical curve it will be enough to apply a central symmetry with respect to the origin.

In this section we will describe the main properties of tropical algebraic curves. First of all how we can deduce the graph Z_B of a T-curve $B \subset \mathbb{R}^2$ starting from its T-polynomial. Let

$$\bigoplus_{(i,j)\in\Delta\cap\mathbb{Z}^2} b_{ij} \odot x^i \odot y^j$$

be the equation of B, where Δ is the triangle of vertices $(0,0), (d,0), (0,d)$. In order to draw the graph, we need to determine the set of points $(x, y) \in \mathbb{R}^2$ such that

$$\max_{(i,j)\in\Delta\cap\mathbb{Z}^2} \{b_{ij} + ix + jy\}$$

is attained twice. We will divide the description of this graph into two parts:

Local behaviour. Observe that the graph of B is a union of segments and half-lines where two of the linear forms attain the same value which is smaller than the one assumed by the remaining forms. This means that we can imagine Z_B as a graph with some of the edges prolonged indefinitely. Let $v \in Z_B$ be a vertex then, after a translation, we can always assume that $v = (0, 0)$. Consider the set of linear forms which attain the maximum value at v:

$$i_1x + j_1y, \ldots, i_rx + j_ry$$

and let $p_k := (i_k, j_k)$ for $k = 1, \ldots, r$.

2.1. *Lemma.* — Let Δ_v be the convex hull of $p_1, \ldots, p_r$, then there exists a neighborhood U of v such that the intersection $U \cap Z_B$ is a graph which is dual to $\partial\Delta_v$.

Proof. Consider the set

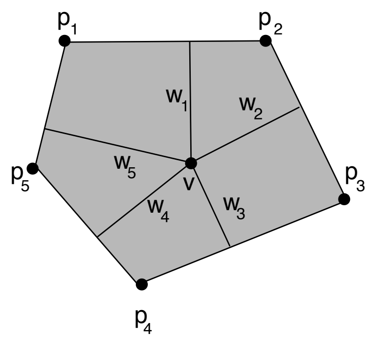
$$H^{sk}_{\geq 0} := \{(x, y) \in \mathbb{R}^2 \mid i_sx + j_sy \geq i_kx + j_ky\}$$

For $k \neq s$ this set is a half plane delimited by the orthogonal line (through v) to $p_k - p_s$ and not containing the point p_k. Now assume that p_s belongs to the interior of Δ_v, then the set of directions $p_k - p_s$, where k moves over all the vertices of δ_v, describes an angle of 2π. This immediately implies that the intersection:

$$\bigcap_{k \neq s} H^{sk}_{\geq 0} = (0,0),$$

so that for any $(x,y) \in \mathbb{R}^2$ there exists $k \in \{1, \dots r\}$, $k \neq s$ such that $i_s x + j_s y \leq i_k x + j_k y$. In a similar way it is possible to prove that only the edges of Δ_v correspond to edges of $U \cap Z_B$. □

Lemma 2.1 shows that the edges of $U \cap Z_B$ are all the orthogonal directions to the edges of Δ_v going out from v.



This local description of Z_B allows us to prove that the dual graph of Z_B is a subdivision of Δ which takes the name of *regular subdivision*. Now let $[w_1], \dots, [w_r]$ be the edges of $U \cap Z_B$ incident to v. Since these edges have rational slopes we can consider the primitive vector w_i defining $[w_i]$.

2.2. *Proposition.* — Let μ_i be the integral length (the number of integer points minus one) of the dual edge of w_i, then:

$$\sum_{v \in [w_i]} \mu_i w_i = 0.$$

Proof. Let $p_1, \dots, p_l$ be a renumbering of the vertices of Δ_v in clockwise order, so that $p_{i+1} - p_i$ is orthogonal to w_i for $i = 1, \dots, l-1$ and $p_1 - p_l$ is orthogonal to w_l. Consider the matrix

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and observe that $M(p_{i+1} - p_i) = \mu_i w_i$ since the right hand side must be a vector orthogonal to $p_{i+1} - p_i$ which is a μ_i multiple of the primitive vector w_i. From the equality

$$\sum_{v \in [w_i]} \mu_i w_i = \sum_{i=1}^{l-1} M(p_{i+1} - p_i) + M(p_1 - p_l)$$

and the linearity of M the first equality follows. □

Global behaviour. An algorithm for determining the regular subdivision of Δ is the following. Given the tropical curve B, consider the polyhedral set so defined:

$$\tilde{\Delta} := \text{Convex hull}\,\{(i,j,b_{ij}) \mid (i,j) \in \Delta \cap \mathbb{Z}^2\}.$$

Define $\tilde{\Delta}_+$ to be the the set of faces in $\tilde{\Delta}$ such that the outward normal direction has the third coordinate non-negative and let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection on the first two coordinates.

2.3. *Lemma.* — Let $q_1, \ldots, q_r \in \mathbb{R}^3$ be a set of distinct points and let $\tilde{\Delta}$ be their convex hull, then the following are equivalent:

(i) q_s belongs to the interior of $\tilde{\Delta}$,

(ii) There exists no plane through q_s such that the r points lie in the same open half space delimited by it.

Proof. For simplicity we can assume that $s = 1$, so that q_1 belongs to the convex hull of $q_2, \ldots, q_r$ if and only if there exists $\epsilon_i \geq 0$ with $\epsilon_2 + \cdots + \epsilon_r = 1$ such that

$$q_1 = \epsilon_2 q_2 + \cdots + \epsilon_r q_r.$$

Let L be a linear form which vanishes on a plane $H \ni q_1$, then

$$0 = \epsilon_2 L(q_2) + \cdots + \epsilon_r L(q_r),$$

so that either $L(q_i) = 0$ for all $i = 2, \ldots, r$ or L attains both a positive and a negative value at some of the points. □

2.4. *Proposition.* — The image of the faces of $\tilde{\Delta}_+$ through π is the regular subdivision of Δ associated to B.

Proof. For any monomial of B, define the corresponding point: $q_s = (i_s, j_s, b_{i_s j_s})$ and observe that, for a given $(x,y) \in \mathbb{R}^3$ we have that:

$$i_s x + j_s y + b_{i_s j_s} \geq i_k x + j_k y + b_{i_k j_k} \quad \Longleftrightarrow \quad \langle q_k - q_s, (x,y,1)\rangle \leq 0.$$

In other words, the linear form associated to q_s realizes the maximum value (between all the linear forms associated to the q_k's) at (x,y) if and only if all the vectors of the form $q_k - q_s$ have a negative projection on $(x,y,1)$. Let H be the plane through q_s which is orthogonal to $(x,y,1)$ and let H_- be the open half space which does not contain the vector $(x,y,1)$. Then the preceding inequality is satisfied if and only if

$$q_k \in H_-.$$

By lemma 2.3 this happens if and only if q_s is a vertex of $\tilde{\Delta}$. Observe that there must exists at least one maximal face $F \subset \tilde{\Delta}$ which contains q_s and whose outward normal vector has a non-negative third coordinate, so that the proposition is proved. □

2.5. **Exercises.**

(1) Complete the details of the last part of the proof of Lemma 2.1.

(2) Write a computer program for drawing tropical plane curves which makes use of the algorithm explained in Proposition 2.4.

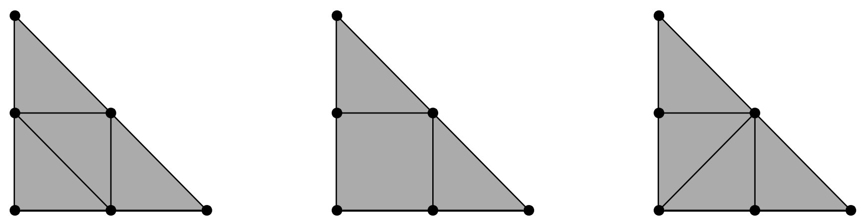
3. Tropical conics and cubics

The results of the previous section can be applied to easily classify all the possible combinatorial types of tropical conics. The triangle Δ associated to

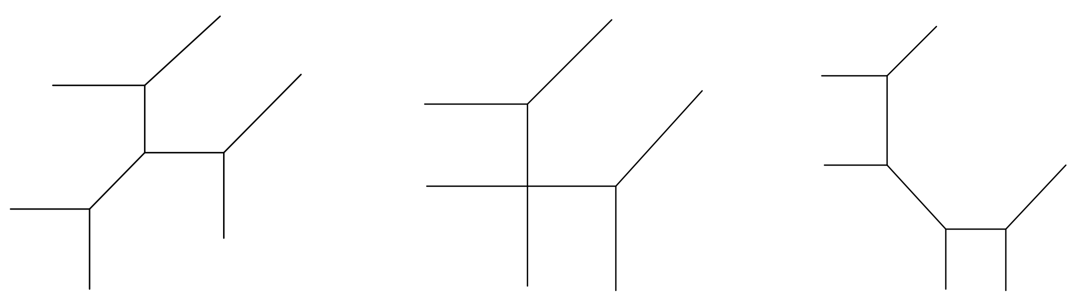
$$b_{20} \odot x^2 \oplus b_{11} \odot x \odot y \oplus b_{02} \odot y^2 \oplus b_{10} \odot x \oplus b_{01} \odot y \oplus b_{00}$$

has vertices: $(0,0), (2,0), (0,2)$. By Proposition 2.2 the regular subdivision of Δ associated to B is dual to the graph of Z_B.

For example consider the subdivisions

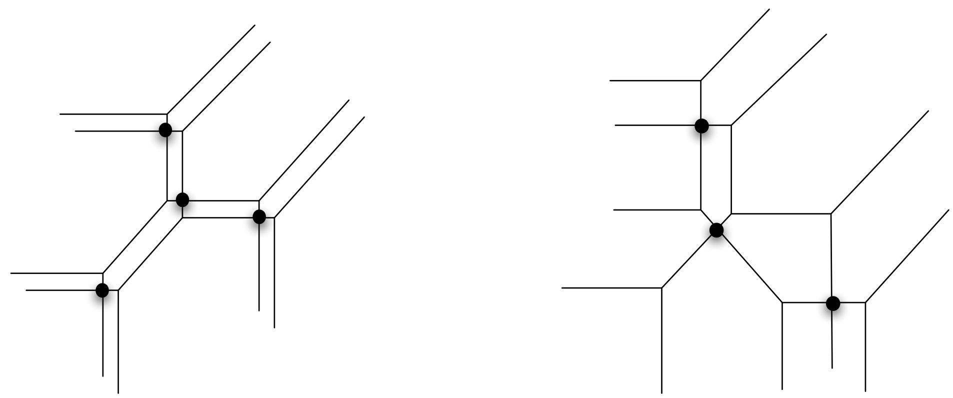


Then the corresponding tropical conics have the following graphs:



Observe that there exist a continuous deformation of the first graph into the third which pass through the second. This deformation corresponds to a one dimensional family of conics which contains the three members. The second conic represents a graph which can be also obtained by drawing two distinct tropical lines. In fact it is possible to prove that this conic is reducible.

The combinatorial types of the intersection of two tropical conics are very numerous, as an example consider the following:

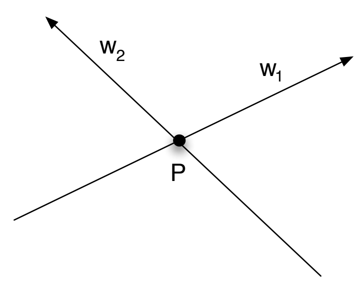


Even if the number of intersection points are different, it is possible to associate to any such $p \in C_1 \cap C_2$ an *intersection multiplicity* m_p so that

$$\sum_{p \in C_1 \cap C_2} m_p = 4.$$

Before defining the intersection multiplicity, observe that we are interested in a *stable number*, this means a number which does not change for small deformations of the two curves (for example by acting on one of the two curves by means of a translation). This immediately implies that we need a definition just for the case of double points, as any other intersection deforms into a union of these.

Now, let U be a sufficiently small neighborhood of the intersection point $p \in C_1 \cap C_2$, so that the two branches can be represented by two lines crossing at p:



These lines have rational slope (since they are obtained as the zero set of two rational linear forms), so that there are two primitive vectors $w_1, w_2 \in \mathbb{Z}^2$ which represent them. Then we have

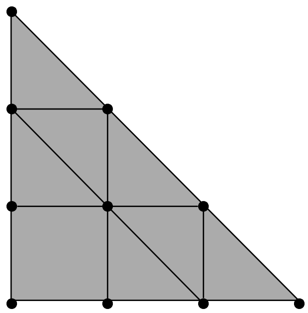
$$m_p := \mu_1 \mu_2 |\mathrm{Det}(w_1, w_2)|.$$

It is evident that the preceding definition of the intersection multiplicity of two tropical curves at a given point does not depends on the fact that C_1 and C_2 are conics but it is well defined for any pair of tropical curves. Moreover, it is easy to associate a *degree* to any tropical curve, just looking at its defining polynomial and defining it as usual. In this way we have the following theorem which is weel known in the classical case:

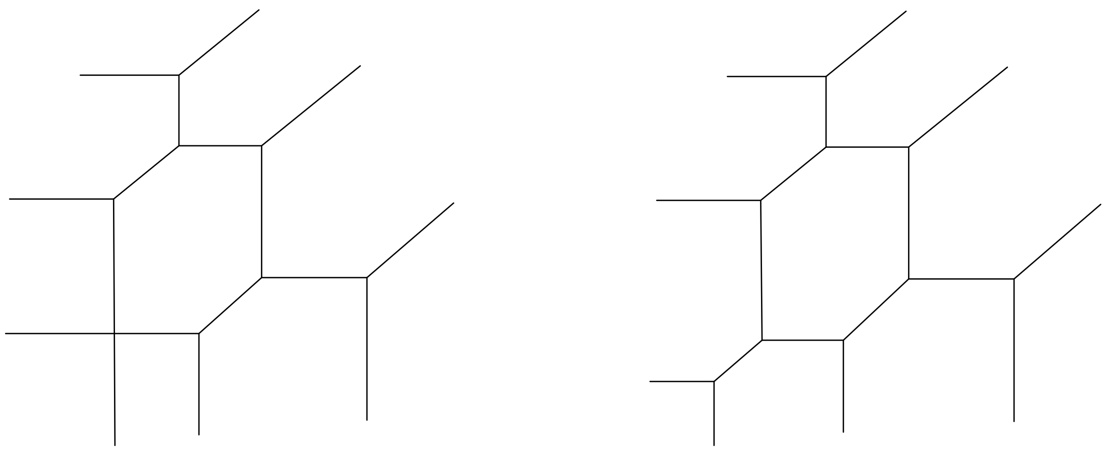
3.1. *Theorem.* — (Bezout) Let B_1, B_2 be two tropical curves of degrees d_1, d_2 respectively. Assume that the intersection of these curves if finite, then the following equality holds:

$$\sum_{p \in B_1 \cap B_2} m_p = d_1 d_2.$$

The reason for considering conics and cubics as examples of tropical plane algebraic curves is that they provide many of the fundamental properties of the general ones. As an example of tropical cubics consider the following regular subdivisions:



The corresponding dual curves are given below:



These curves have a new property with respect to conics: they contain a loop. The main difference between them is that the second curve is a trivalent graph while the first one has a point of valence 4. We have already seen that a conic with a point of valence 4 is reducible (it is the union of two tropical lines). This is no longer true for tropical cubics. In fact both the cubics of the picture are irreducible, and the first is *singular* along the point of valence 4. This means that it is possible to prove that this curve is the image of a singular curve (and only of a singular one) of $(K^*)^2$. To any tropical plane curve it is possible to associate an abstract trivalent graph Γ by separating the edges incident along points of valence ≥ 4. Doing this for the first cubic, one obtain a graph which is a tree. This corresponds to the fact that this graph represents a genus 0 curve. In this way it is possible to define the *genus* of a tropical plane curve as the dimension of $H_1(\Gamma, \mathbb{Z})$.

3.2. **Exercises.**

(1) Prove the Bezout theorem for the intersection of two tropical conics.

(2) Find the maximal multiplicity μ for an edge of a tropical conic and a tropical cubic.

(3) Determine the maximal intersection multiplicity of two tropical plane cubics at a given point.

(4) Prove that a tropical cubic with two points of valence 4 is reducible.

(5) Find a necessary and sufficient condition for a tropical cubic in order to be reducible.

References

[1] Gathman, A. *Tropical algebraic geometry* math.AG/0601322

[2] Richter-Gebert, J. Sturmfels, B. and Theobald, T. *First steps in tropical geometry.* math.AG/0306366.

DEPT. OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA
E-mail address: alaface@mast.queensu.ca