INTRODUCTION TO TROPICAL GEOMETRY

ANTONIO LAFACE

Introduction

The aim of these notes is to introduce the subject of tropical geometry and subsequently describe the correspondence between algebraic varieties and tropical varieties.

In section 1 we introduce amoebas as images of algebraic varieties through the Log map. After discussing their main properties we will focus our attention on amoebas defined over non-archimedean fields and their properties.

1. Amoebas

Let us consider the map $\mathrm{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n$ defined as

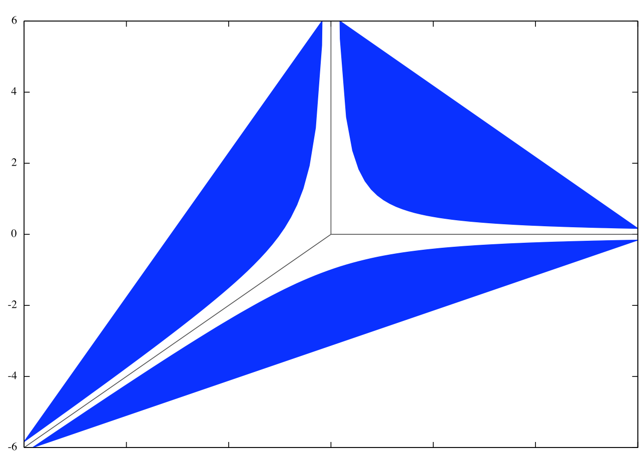
$$\mathrm{Log}(z_1, \ldots, z_n) := (\log |z_1|, \ldots, \log |z_n|).$$

Given an algebraic variety $X \subset (\mathbb{C}^*)^n$ we define its *amoeba* to be the set:

$$\mathcal{A}(X) := \mathrm{Log}(X).$$

Amoebas of algebraic varieties were first introduced for studying topological properties of real algebraic curves. In other words, given a real algebraic curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ one can ask about the possible topological types of the pairs $(C, \mathbb{P}^2_{\mathbb{R}})$. This problem can be approached by using amoebas.

1.1. *Example.* — Consider the line $L \subset (\mathbb{C}^*)^2$ of equation $x + y + 1 = 0$, then $-\mathcal{A}(X)$ is given by the white drawing inside the blue region:



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The fact that the amoeba of L has three branches which go to infinity depends on the fact one of the coordinates $(z_1, z_2) \in L$ can be close to 0 or both can be close to ∞. It is an easy exercise to determine the directions of the asymptotic lines of the amoeba.

The following theorem allows to understand better the shape of an amoeba X. Assume that X is a zero set of a polynomial $f(\mathbf{z}) = \sum a_{\mathbf{i}}\mathbf{z}^{\mathbf{i}}$ and let Δ_f be the its Newton polytope:

$$\Delta_f := \text{Convex hull}\,\{\mathbf{i} \in \mathbb{Z}^n \mid a_{\mathbf{i}} \neq 0\}.$$

1.2. *Theorem.* — (See [1]) There exists a locally constant function

$$\mathrm{ind} : \mathbb{R}^n \setminus \mathcal{A}(X) \to \Delta_f \cap \mathbb{Z}^n$$

which maps distinct components of the complement of $\mathcal{A}(x)$ to distinct points of Δ_f.

In order to give an idea of the proof of the preceding theorem consider the map $\log|f| : \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ which is a plurisubharmonic function. Recall that a function F in a domain $\Omega \subset \mathbb{C}^n$ is called plurisubharmonic if its restriction to any complex line L is subharmonic. Let $N_f : \mathbb{R}^n \to \mathbb{R}$ be the average of $\log|f|$ along the fibers of the Log map:

$$N_f(\mathbf{x}) = \frac{1}{(2\pi)^n}\int_T \log|f(e^{\mathbf{x}+i\theta})|d\mu,$$

where $T = \mathrm{Log}^{-1}(\mathbf{x})$ is the torus with measure $d\mu = d\theta_1 \wedge \ldots \wedge d\theta_n$. This function is called the Ronkin function. It is possible to prove that it takes real (finite) values even over $\mathcal{A}(X)$ where the integral is singular.

1.3. *Proposition.* — (See [1]) The function N_f has the following properties:

(i) it is convex,
(ii) it is strictly convex over $\mathcal{A}(X)$
(iii) it is linear over each component of $\mathbb{R}^n \setminus \mathcal{A}(X)$

Idea of the Proof. The main step of the proof depends the fact that $\log|f| : (\mathbb{C}^*)^n \to \mathbb{R}$ is plurisubharmonic. This in turn implies that $\Delta N_f(\mathbf{x}) \geq 0$ for any $\mathbf{x} \in \mathbb{R}^n$ and this is equivalent to say that N_f is a convex function.

Moreover, if $\mathbf{x} \in \mathbb{R}^n \setminus \mathcal{A}(X)$ one has that N_f is pluriharmonic in x and this simply means that it is a linear function, since the second derivative of N_f along any the direction of a line $R \subset \mathbb{R}^n$ vanishes. $\square$

The preceding proposition implies that the gradient ∇N_f is constant over each component $E \subseteq \mathbb{R}^n \setminus \mathcal{A}(X)$. Moreover, by Jensen's formula we have that:

$$\nabla N_f(E) \in \mathbb{Z}^n \cap \Delta_f.$$

As a consequence of the last discussion, it is possible to define

$$\mathrm{ind}(x) := \nabla N_f(x)$$

for each $x \in \mathbb{R}^n \setminus \mathcal{A}(X)$. The function ind constructed in this way has the required properties.

1.4. *Example.* — Going back to the line of equation $x + y + 1 = 0$, its Newton polytope Δ_f is the convex hull of the three points $(0,0), (1,0), (0,1)$. In this case the three points represent the connected components of $\mathbb{R}^2 \setminus \mathcal{A}(X)$.

1.5. *Example.* — Consider the polynomial

$$f(x,y) = xy - x + y + 1,$$

then Δ_f is the square of vertices $(0,0), (1,0), (0,1), (1,1)$. It is not difficult to see that $\mathcal{A}(X)$ has four branches which go to infinity corresponding to the points $(1,0), (0,-1)$ and the two asymptotes of equation $x = -1$ and $y = 1$.

The spine of an amoeba. Looking at the graph of $-\mathcal{A}(L)$, where L is the line of equation $x + y + 1 = 0$, it is possible to see that inside it there is a tropical line which represent the three asymptotic directions of the branches of the amoeba. This tropical variety is called *the spine* of the amoeba.

A precise definition of the spine of an amoeba of an hypersurface $X \subset (\mathbb{C}^*)^n$ of equation $f = 0$, is given in terms of its Ronkin function. Consider the function:

$$N_f^\infty = \max_E N_E,$$

where E runs over all components of $\mathbb{R}^n \setminus \mathcal{A}(X)$ and N_E is the linear function obtained by extending $N_f|_E$ to $\mathbb{R}^n$ by linearity.

The spine $\mathcal{S}_X$ of the amoeba $\mathcal{A}(X)$ is the set of points in $\mathbb{R}^n$ where N_f^∞ is not locally linear. The definition immediately implies that $\mathcal{S}_X \subset \mathcal{A}(X)$ and moreover that $\mathcal{S}_X$ is a piecewise-linear polyhedral complex.

It is possible to prove that the spine $\mathcal{S}_X$ is a strong deformational retract of the amoeba $\mathcal{A}(X)$, so that each component of $\mathbb{R}^n \setminus \mathcal{S}_X$ contains a unique component of $\mathbb{R}^n \setminus \mathcal{A}(X)$.

1.6. **Exercises.**

(1) Let a, b, c be three non-zero complex numbers and let $L \subset (\mathbb{C}^*)^2$ be the line of equation $ax + by + c = 0$.
   (i) Determine the amoeba $\mathcal{A}(L)$.
   (ii) Find the spine $\mathcal{S}_X$ of $\mathcal{A}(L)$ and determine the tropical equation of $-\mathcal{S}_X$.

(2) Determine the amoebas of the following conics:
   (i) $x^2 + y^2 - 1 = 0$
   (ii) $x^2 + y^2 + 1 = 0$
   (iii) $xy + x - y + 1 = 0$

   (Hint: find a rational parametrization of these conics).

2. NON ARCHIMEDEAN AMOEBAS

Let K be a field. We start this section by recalling that there is a one to one correspondence between *non-archimedean* norms and *valuations* defined over K. The axioms for a non-archimedean norm on K are compared to the valuation ones:

1. $\lvert x\rvert = 0 \Leftrightarrow x = 0$,	1′. $v(x) = +\infty \Leftrightarrow a = 0$
2. $\lvert xy\rvert = \lvert x\rvert\lvert y\rvert$,	2′. $v(xy) = v(x) + v(y)$
3. $\lvert x+y\rvert \leq \max\{\lvert x\rvert, \lvert y\rvert\}$	3′. $v(x+y) \geq \min\{v(x), v(y)\}$

and the correspondence between the two is realized by the map:

$$v(x) := -\log|x|.$$

for any $x \in K$. In what follows we will assume that K is an algebraically closed field which is a complete metric space with respect to the metric induced by the norm. Let us consider the map $\mathrm{val} : (K^*)^n \to \mathbb{R}^n$ defined as

$$\mathrm{val}(x_1, \ldots, x_n) := (v(x_1), \ldots, v(x_n)).$$

Given an algebraic variety $X \subset (K^*)^n$ we define its tropical variety $T(X) \subset \mathbb{R}^n$ to be the set:

$$T(X) := \overline{\mathrm{val}(X)}.$$

Observe that the tropical variety associated to X is defined as the negative of the amoeba of Section 1. The other main difference here is represented by the non-archimedean property. In this way the $-\mathrm{Log}$ map is realized by means of a valuation. This will allow us to give an easier description of $T(X)$ with respect to the complex case.

2.1. *Example.* — Let $\mathbb{C}((t))$ be the field of formal Laurent series, i.e. series of the form $\sum_{i=r}^{\infty} f_i t^i$, with $f_i \in \mathbb{C}$ and $r \in \mathbb{Z}$. The algebraic closure of this field is given by:

$$\overline{\mathbb{C}((t))} = \bigcup_{n\geq 1} \mathbb{C}((t^{1/n})).$$

The field $\overline{\mathbb{C}((t))}$ is equipped with a $\mathbb{Q}$-valued valuation: $\mathrm{ord}(\sum f_\alpha t^\alpha) := \min\{\alpha \mid f_\alpha \neq 0\}$.

Let K be an algebraically closed field of characteristic 0 which is endowed with a non-archimedean norm and consider the hypersurface $X \subset (K^*)^n$ of equation $f(\mathbf{z}) = 0$, where

$$f(\mathbf{z}) = \sum_{\mathbf{i}\in\mathbb{Z}^n} a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}},$$

with coefficients $a_{\mathbf{i}} \in K$. For any $\mathbf{u} \in \mathbb{R}^n$ denote by

$$f^{\mathrm{T}}(\mathbf{x}) = \bigoplus_{\mathbf{i}\in\mathbb{Z}^n} v(a_{\mathbf{i}}) \odot \mathbf{x}^{\mathbf{i}},$$

and let $Z(f^T)$ be the tropical variety associated to f^T, which means the set of points where the minimum is attained at least twice.

Before going on, recall that the extended Newton polytope of $f(z)$ is defined as:

$$\tilde{\Delta}_f := \text{Convex hull } \{(i, u) \in \mathbb{Z}^n \times \mathbb{R} \mid u \geq v(a_i)\} \subset \mathbb{R}^{n+1}$$

2.2. *Lemma.* — Let $t_1, \dots, t_r \in K^*$ be such that $t_1 + \cdots + t_r = 0$, then $\max\{|t_1|, \dots, |t_r|\}$ is attained at least twice.

Proof. First of all observe that $|-t| = |t|$ for all the elements $t \in K$. Now, assume for simplicity that the maximum is attained in t_1, then the equation $t_1 = -(t_2 + \cdots + t_r)$ gives $|t_1| = |t_2 + \cdots + t_r| \le \max\{|t_2|, \dots, |t_r|\}$ and this implies the thesis. $\square$

2.3. *Lemma.* — (See [2]) Let $g(t) = \sum_{i=r}^{s} a_i t^i$ be a Laurent polynomial and let $\tilde{\Delta}_g$ be its generalized Newton polytope, then the valuations of the $s - r$ roots of g are the negatives of the edge slopes of $\tilde{\Delta}_g$.

2.4. *Theorem.* — (See [2]) Given $X \subset (K^*)^n$ defined by $\{f = 0\}$ the following holds:

$$T(X) = Z(f^T).$$

Proof. In what follows we will denote by T, Z the two sets $T(X), Z(f^T)$. Our first step will be to prove that:

(i) $T \subseteq Z$.

Since Z is closed, it is enough to show that $\mathrm{val}(X) \subset Z$. Let $\mathbf{x} \in \mathrm{val}(X)$, then there exists $\mathbf{z} \in K^n$ such that $v(\mathbf{z}) = \mathbf{x}$. Since $f(\mathbf{z}) = \sum a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} = 0$, it follows by lemma 2.2 that there are at least two terms in the sum where the norm attains its maximum. This implies that $\min\{v(a_{\mathbf{i}} \mathbf{z}^{\mathbf{i}})\}$ or equivalently $\min\{v(a_{\mathbf{i}}) + \mathbf{x} \cdot \mathbf{i}\}$ is attained twice, so that $\mathbf{x} \in Z$.

(ii) $Z \subseteq T$.

Let $\mathbf{x} \in Z$, then without loss of generality we can assume that there exists $\mathbf{c} \in (K^*)^n$ such that $\mathrm{val}(\mathbf{c}) = \mathbf{x}$ (the image of val is dense in $\mathbb{R}^n$ and Z is a polyhedral rational on $\mathrm{val}(K^*)$). The effect of a change of variables of type $z_i \mapsto z_i \cdot c_i$ is to translate both T and Z by the vector $-\mathrm{val}(\mathbf{c}) = -\mathbf{x}$. After this translation we have that

$$\mathbf{0} \in Z,$$

so that it is enough to prove that $\mathbf{0} \in T$. We will find a root $\mathbf{z}^0$ of f of the form

$$z_i^0 = (t_0)^{b_i}, \quad t_0 \in K^*, \quad v(t_0) = 0$$

for an appropriate choice of $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{Z}^n$. Indeed, let

$$f_{\mathbf{b}}(t) = f(t^{b_1}, \dots, t^{b_n}) = \sum a_{\mathbf{i}} t^{\mathbf{b} \cdot \mathbf{i}} \in K[t, t^{-1}].$$

The fact that $\mathbf{0} \in Z$ means that $\tilde{\Delta}_f$ has a face F of positive dimension which is horizontal (we assume F to be maximal with this property). Assume that $\mathbf{b} \in \mathbb{Z}^n$ is generic in the following sense: for each edge $[(\mathbf{m}, u), (\mathbf{i}, u)]$ of F we have $\mathbf{b} \cdot (\mathbf{m} - \mathbf{i}) \neq 0$. Then $\tilde{\Delta}_{f_{\mathbf{b}}}$ has a horizontal edge and by lemma 2.3 we have that $f_{\mathbf{b}}$ has a root t_0 with $v(t_0) = 0$.

$\square$

2.5. **Exercises.**

(1) Assume that $K = \overline{\mathbb{C}((t))}$ and determine the amoeba of $x + y + 1 = 0$.

(2) Assume that $a_{11}x^2 + 2a_{12}xy + a_{22}y^y + 2a_{13}x + 2a_{23}y + a_{33} = 0$ is the equation of a smooth conic defined over a non-archimedean field K. Determine all the possible combinatorial types of its amoeba.

3. APPENDIX

In this section we recall some important definitions used in the preceding pages.

3.1. **Plurisubharmonic functions.** A function

$$f : \Omega \to \mathbb{R} \cup \{-\infty\},$$

where $\Omega \subseteq \mathbb{C}^n$ is a domain, is called *plurisubharmonic*, if it is upper semi-continuous, and for every complex line $L \subset \mathbb{C}^n$, the restriction f_L is a *subharmonic function*, i.e.

$$f_L(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f_L(z + re^{i\theta}) d\theta$$

for any complex number z and positive real r. It is possible to prove that if $g(z)$ is a holomorphic function, then $\log |g(z)|$ is plurisubharmonic.

3.2. **Jensen's formula.** Let $f : \mathbb{C} \to \mathbb{C}$ be an holomorphic function such that $f(0) \neq 0$. Assume that $f(z)$ has no zeroes on the circumference $|z| = e^x$, then the following formula holds:

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{x+i\theta})| d\theta = nx + \log |f(0)| - \sum_{k=1}^{n} \log |a_k|,$$

where $a_1, \dots, a_n$ are the zeroes of f in $|z| < e^x$.

REFERENCES

[1] Mikhalkin, G. *Amoebas of algebraic varieties and tropical geometry.* math.AG/0403015
[2] Einsiedler, M., Kapranov, M., Lind, D. *Non-archimedean amoebas and tropical varieties* math.AG/0408311

DEPT. OF MATHEMATICS AND STATISTICS, QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA
E-mail address: `alaface@mast.queensu.ca`