EUROPEAN JOURNAL OF ECONOMICS, FINANCE AND ADMINISTRATIVE SCIENCES

ISSN: 1450-2275

Issue 10
March, 2008

EUROPEAN JOURNAL OF ECONOMICS, FINANCE AND ADMINISTRATIVE SCIENCES
http://www.eurojournals.com/EJEFAS.htm

Indexing / Abstracting
European Journal of Economics, Finance and Administrative Sciences is indexed in Scopus, Elsevier Bibliographic Databases, EMBASE, Ulrich, DOAJ, Cabell, Compendex, GEOBASE, and Mosby.

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European Journal of Economics, Finance and Administrative Sciences is published in the United States of America at Lulu Press, Inc (Morrisville, North Carolina) by EuroJournals, Inc.

European Journal of Economics, Finance and Administrative Sciences
ISSN 1450-2887 Issue 10 (2008)

http://www.eurojournalsn.com

A Substitute for Riskless Assets and the Market Portfolio

Jose Rigoberto Parada Daza
Professor of Finances at the Universidad de Concepcion- Chile
Department of Administration, Economics
And Administrative Sciences, Casilla 160-C
Tel: (56 41) 20 4172; Fax: (56 41) 204172
E-mail: rparada@udec.cl

Abstract

A review of the traditional literature of the efficient frontier of mean-variance (MV) reveals the development of an alternative to CAPM and the Capital Market Line, the re-elaboration of a test to measure the efficient frontier without riskless assets, and the development of the efficient frontier with short sales. These perspectives let us re-elaborate the concept of a risky substitute portfolio that generates a profit equal to that of a riskless financial asset. This papers develops some propositions for the formation of a portfolio that can substitute for the riskless asset, but that is made up of risky assets and has set investment proportions. Likewise, the paper analyzes the creation of a substitute for the market portfolio.

Keywords: Riskless asset, CAPM, mean-variance, efficient frontier

1. Introduction

The CAPM model is based on what is called the Security Market Line, and is represented by:

$$E(R_i) = R_F + \beta_i \left[E(R_m) - R_F \right] \quad \forall\, i \in M$$

where $\beta_i = cov(R_i,R_m)/var(R_m)$, M = Security Market. β_i is the Beta of asset i, $E(R_m)$ is the market portfolio return, and R_F = the riskless interest rate.

We develop the Capital Market Line by investing α% of our resources in the market portfolio and (1-α)% in a riskless asset; the return of this new asset is:

$R_i = (1 - \alpha)R_F + \alpha E(R_m)$. Finally, in equilibirum, the Global Model is as follows:

$E(R_i) = R_F + (\sigma_i/\sigma_m)\,(E(R_m) - R_F)$ where

σ_i = standard deviation of an asset's return, and

σ_m = standard deviation of the market portfolio's return.

In an early work, Black [1972] demonstrated that the model is verified even without a riskless asset, or one having $\beta = 0$. The model defines market portfolios as the percentage of risky assets j, for j = 1,… k and having $\beta = 1$ so that it satisfies the following relationship according to Jarrow, [1988]:

$$P_m = \left[\sum \tilde{N}_j^i \right] p(x_j) / \sum_{k=1}^{k} \left[\sum \bar{x}_k^0 \right] p(x_k)$$

$$where \quad \sum_{j=1}^{k} P_{mj} = 1$$

Thus, CAPM requires a riskless asset ($\beta = 0$) as well as risky assets from the market portfolio ($\beta = 1$). With this article, we will show how a portfolio with a different definition of riskless and market portfolios, and which can be substituted by other equivalent portfolios, can exist.

2. Substitute Portfolio of Riskless Assets

Using Markowitz's [1958] definition of the efficient frontier of mean-variance (MV) Jarrow [1988] lays out an alternative to CAPM and the Capital Market Line. He then offers us an interesting perspective by using this same MV methodology to re-elaborate the Capital Market Line using non-linear optimization. Kandel [1984] elaborates a test to measure the efficient frontier without riskless assets. Elton and Gruber [1995] develop the case of the efficient frontier based on short sales and in which classic methods of determination are described. We use these perspectives to re-elaborate the concept of a substitute portfolio whose return is equal to the return of a riskless portfolio. Since our methodology is based on the development of Sharpe's initial Market Model [1964, 1970], we use the following assumptions and propositions.

Assumptions

1. Suppose that CAPM and MV assumptions are met, and that there is a possibility for short sales.
2. There are two risky assets (1 and 2) that have the following conditions:
   - 2.1) These assets have the following returns when in equilibrium, i.e. lie along the Security Market Line (SML), (Jarrow, pg. 223)

     $E(R_1) = R_F + \beta_1 [E(R_m) - R_F]$

     $E(R_2) = R_F + \beta_2 [E(R_m) - R_F]$

     In general: $E(R_i) = R_F + \beta_i [E(R_m) - R_F]$
   - 2.2) When in equilibrium, i.e lie along SML, their total risks are given by (Sharpe,1970):

     $\sigma_1^2 = \beta_1^2\sigma_m^2 \quad and \quad \sigma_2^2 = \beta_2^2\sigma_m^2$

     In general: $\sigma_i^2 = (\beta_i\sigma_m)^2$ = systematic risk.
   - 2.3) When assets 1 and 2 are in equilibrium, i.e. lie along SML, the covariances of these two assets in function of the Betas of the CAPM model are as follows: (See Appendix 1)

     $\sigma(1,2) = (\beta_2/\beta_1)/\sigma^2_1 = (\beta_1\beta_2\sigma_m^2)$

     where σ^2_m = market portfolio variance.

     In general: $\sigma (i,j) = (\beta_i/\beta_j)/\sigma_j^2$

Proposition I. By mixing risky assets, a portfolio can be made that generates a return equivalent to the return of a riskless asset and which has zero risk. This portfolio includes a risky asset and is financed with personal equity and debt or short sales.

Proof

We can create a portfolio with these two assets by investing x_1 and x_2 so that $x_1 + x_2 = 1$; the risk, σ^2_c, can be minimized following the MV model, in other words:

$$\sigma_c^2 = (\beta_1\sigma_m)^2 x_1^2 + (1 - x_1)^2(\beta_2\sigma_m)^2 + 2x_1(1 - x_1)(\beta_1\beta_2)\sigma_m^2 \tag{1}$$

By resolving the first order conditions $\partial\sigma^2_c/\partial x_1=0$ and performing algebraic operations, the following proportions are obtained: (See Appendix 2)

$$x^*_1 = \beta_2/(\beta_2 - \beta_1) \tag{2}$$

$$x^*_2 = -\beta_1/(\beta_2 - \beta_1) \tag{3}$$

A portfolio having the proportions x^*_1 and x^*_2 has the following characteristics in expected return, $E(R_c)$, and risk, σ^2_c:

$$E(R_c) = \left[R_F + \beta_1(E(R_m) - R_F)\right]\left(\frac{\beta_2}{\beta_2 - \beta_1}\right) - \left[R_F + \beta_2(E(R_m) - R_F)\right]\left(\frac{\beta_1}{\beta_2 - \beta_1}\right)$$

By rearranging, it can be shown that:

$E(R_c) = R_F$ and

$$\sigma_c^2 = (\beta_1\sigma_m)^2\left[\beta_2/(\beta_2 - \beta_1)\right]^2 + (\beta_2\sigma_M)^2\left[-\beta_1/(\beta_2 - \beta_1)\right]^2 - 2\left[(\beta_1\beta_2)/(\beta_2 - \beta_1)^2\right](\beta_1\beta_2\sigma_M^2)$$

By algebraic reduction, it can be shown that: $\sigma^2_c = 0$

Therefore, a portfolio investing x^*_1 and x^*_2 in two assets has an expected return of R_F and a risk equal to $\sigma^2_c = 0$. In other words, this portfolio generates returns equivalent to a riskless rate and with zero risk, which is the same as investing all our personal equity in a riskless asset.

Investing and Financing Analysis of new portfolio

From x^*_1 and x^*_2, it can be seen that, if $x^*_1>1 \Rightarrow x^*_2<0$ or if $x^*_1<0 \Rightarrow x^*2>1$. So for any $x^*_1>1$, a greater proportion of the resources available will be invested, being summarized as: $x^*_1>1 \Rightarrow x^*_2<0$ or $x_2^*<0 \Rightarrow x^*_2>1$

We know that $x_1 + x_2 = 1$, and if $x^*_1 > 1$ and $x^*_2 < 0$, then: $x^*1 = x^*2 + 1$; this indicates investment in the risky asset x^*_1 and financing x^*_2 with debt (or short sales) and one unit of personal equity. In this case, x^*_2 indicates the times that we must go into debt (or make short sales) per unit of personal equity.

Supposing that we have $C, then the equation becomes the following:

$x^*_1C = x^*_2C + C$, or investment = financing

Summarizing:

Investment: x^*_1

Financing:

Debt (or short sale): x^*_2

Personal equity: 1

The debt (or short sale) is made at a risky rate of: $R_F + \beta_2[E(R_m) - R_F]$; the investment in asset 1 generates a return of: $R_F + \beta_1[E(R_m) - R_F]$. Supposing that we can invest a total of $C of personal equity, then the profit to be invested in a riskless asset is $\$R_FC$.

A riskless portfolio that is equivalent to investing in a riskless asset will be made up as follows:

		Proportion	**Value**
Investment		x^*_1	x^*_1C
Financing	Debt or short sale	x^*_2	x^*_2C
	Personal equity	1	C

Asset 1 Investment Earnings = $x^*_1[R_F + \beta_1(E(R_m) - R_F)]$

Financing Cost = $x^*_2[R_F + \beta_2(E(R_m) - R_F)]$

By replacing the values of x^*_1 and x^*_2 with $x^*_1 = \beta_1/(\beta_2 - \beta_1)$ and $x^*_2 = -\beta_2/(\beta_2 - \beta_1)$, we arrive at: Net Utility = Investment Earning – Financing Cost = R_F

A particular case of x^*_1 and x^*_2, that is deduced from equations (2) and (3) is that, if asset i has $\beta_i = 0$, then $x^*_1 = 1$, which is the very definition of a riskless asset and coincides with the minimum point of the Capital Market Line. This first proposition shows that an asset of $\beta_i=0$ is not vital. It can be replaced by an alternative portfolio that invests x^*_1 in risky assets, finances them with debt or short sales plus personal equity, and can generate the same riskless return with null risk.

3. A Substitute for the Market Portfolio

Proposition II. According to the Security Market Line, alternative market portfolios, offering the same risk and return as a market portfolio with $\beta = 1$, can be made with risky assets whose Betas are not equal to one.

Proof

a) Alternative market portfolios

The same assumptions hold as in Proposition I. In this case, it is necessary that:

$x_1E(R_1) + (1 - x_1)E(R_2) = E(R_m)$, where $E(R_i)$ = expected return of risky asset i when in equilibrium according to CAPM, $E(R_m)$ = expected return of a market portfolio, and xi = proportion to invest in risky asset i, having a known β_i.

Accordingly, a new portfolio is attempted that has the same expected market return risk, mathematically implying the following:

$$MIN\ L = x_1^2(\beta_1\sigma_m)^2 + (1-x_1)^2(\beta_2\sigma_m)^2 + 2x_1(1-x_1)\beta_1\beta_2\sigma_m^2 + \lambda\left[E(R_m) - x_1E(R_1) - (1-x_1)E(R_2)\right]$$

By solving for $\partial\sigma^2_c/\partial x_1 = 0$ and $\partial\sigma^2_c/\partial\lambda = 0$, and by rearranging algebraically, the following results are obtained for x_1 and x_2: (See Appendix No. 3)

$$x^*_1 = (\beta_2 - 1)/(\beta_2 - \beta_1) \qquad (4)$$

$$x^*_2 = (1 - \beta_1)/(\beta_2 - \beta_1) \qquad (5)$$

We can arrive at a return of E(Rc) and a risk of σ^2_c for a portfolio formed with x^*_1 and x^*_2:

$$E(R_c) = \left[R_F + \beta_1(E(R_m) - R_F)\right](\beta_2 - 1)/(\beta_2 - \beta_1) + \left[R_F + \beta_2(E(R_m) - R_F)\right](1 - \beta_1)/(\beta_2 - \beta_1)$$

By arranging, we arrive at: $E(R_c) = E(R_m)$ and

$$\sigma_c^2 = \left(\frac{\beta_2 - 1}{\beta_2 - \beta_1}\right)^2(\beta_1\sigma_m)^2 + \left(\frac{1-\beta_1}{\beta_2 - \beta_1}\right)^2(\beta_2\sigma_m)^2 + 2\left(\frac{\beta_2 - 1}{\beta_2 - \beta_1}\right)\left(\frac{1-\beta_1}{\beta_2 - \beta_1}\right)(\beta_1\beta_2\sigma_m^2)$$

By arranging, we arrive at: $\sigma^2_c = \sigma^2_m$

In other words, the new portfolio formed of x^*_1 and x^*_2 has an expected return equal to the return of a Market Portfolio, $E(R_m)$, and the same risk, σ^2_m, as this Portfolio.

We can see from (5) and (6) that, if $\beta_i = 1$ in any case, all resources should be invested in this asset. The situation used for the Security Market Line is a specific case of (5) and (6), in which there is an exact coincidence of this line with the efficient frontier

b) Investment and Financing of the Alternative Portfolio

b.1. Investment and personal equity

If $0 < x^*_1 < 1 \Rightarrow 0 < x^*_2 < 1$, when $\beta_2 > \beta_1$, $\beta_2 > 1$, and $1 < \beta_1 < \beta_2$

In this case, we invest in x^*_1 and x^*_2, and finance everything with personal equity. If we have \$C of personal equity, then the investment in asset 1 is: $[(\beta_2 - 1)/(\beta_2 - \beta_1)]C$ and in asset 2 is: $[(1 - \beta_1)/(\beta_2 - \beta_1)]C$ with an expected return of $E(R_c) = E(R_m)$ and a total risk of $\sigma^2_c = \sigma^2_m$. In other words, the return is equal to that of a market portfolio and the risk is equivalent to that of a market portfolio, σ^2_m.

b.2. Investment and financing with debt (or short sales) plus personal equity.

We may see the following situations, depending on each asset's Beta value:

$x^*_1 > 1 \Rightarrow x^*_2 < 0$ or $x^*_1 < 0 \Rightarrow x^*_2 > 1$

In each of these two cases, we obtain a portfolio that has the following relationships:

$x^*_1 = x^*_2 + 1$	if	$x^*_1 > 1$ and $x^*_2 < 0$
or $x^*_2 = x^*_1 + 1$	if	$x^*_1 < 0$ and $x^*_2 > 1$

In general, if: $x^*_i = x^*_j + 1$, then a proportion of x^*_i is invested in asset i, generating a return equal to $R_F + \beta_i [E(R_m) - R_F]$. To finance this investment, we use one unit of personal equity plus debt or short sales in proportion x^*_j at a cost equal to $R_F + \beta_j\,[E(R_m) - R_F]$. This portfolio generates a return of $E(R_m)$ with a risk equal to σ^2_m

In effect: by substituting the values of x^*_1 and x^*_2 in the return function of the new portfolio, we arrive at:

$$E(R_c) = \left(\frac{\beta_2 - 1}{\beta_2 - \beta_1}\right)\left[R_F + \beta_1(E(R_m) - R_F)\right] + \left(\frac{1 - \beta_1}{\beta_2 - \beta_1}\right)\left[R_F + \beta_2(E(R_m) - R_i)\right]$$

By arranging, we arrive at: $E(R_c) = E(R_m)$. Regarding the risk, we arrive at: $\sigma^2_c = \sigma^2_m$

Thus, we show how a portfolio including an investment in asset 1, financed with debt and personal equity, has a risk of σ^2_m and a return of $E(R_m)$, which is equivalent to investing personal equity alone in a market portfolio with $\beta = 1$.

4. The Efficient Frontier and Capital Market Line. A General Case

Both previously mentioned propositions are specific cases of a more general situation determining alternative portfolios having any Beta other than zero or one, the respective values initially established for riskless and market portfolios. In a more general formulation, Brennan [1971] focused on the formation of portfolios with debt and borrowing at different interest rates by using the concept of the efficient frontier. In an analogous approach, Jarrow [1988] focused on the efficient frontier and its relationship with CAPM.

In this paper, we sought out alternative portfolios using Jarrow's and Sharpe's methodologies, since both authors demonstrate market portfolios to be in the efficient frontier, and, therefore, efficient portfolios. However, in general terms, the following question can be asked: Is there a tangency point with the Capital Market Line for any efficient portfolio (or any portfolio within the efficient frontier)? The answer, according to the methodological scheme of propositions I and II, is yes. The proof is as follows:

Our idea is to use two assets, having $\beta \neq 0$ and $\beta \neq 1$, to form a portfolio subject to desired return, R_d. According to Lagrange's conditions, the function to be minimized is:

$$L = x_1^2(\beta_1\sigma_m)^2 + (1 - x_1)^2(\beta_2\sigma_m)^2 + 2x_1(1 - x_2)\beta_1\beta_2\sigma_m^2 + \lambda\left[R_d - x_1R_1 - (1 - x_1)R_2\right]$$

where the same assumptions hold as in propositions I and II.

Using the First Order minimization conditions, we arrive at:

$$\partial L/\partial x_1 = 2x_1\sigma_m^2[\beta_2 - \beta_1]^2 + \lambda[R_2 - R_1] - 2\sigma_m^2\beta_2(\beta_2 - \beta_1) = 0$$

$$\partial L/\partial \lambda = R_d - x_1R_1 + x_1R_2 - R_2 = 0$$

By solving for the two equations simultaneously, the results are the following: (See Appendix No. 4)

$$x_1^{*} = \frac{\beta_2}{\beta_2 - \beta_1} - \frac{(R_d - R_F)}{(R_m - R_F)}\,\frac{1}{(\beta_2 - \beta_1)} \qquad \text{with } \beta_1 \neq \beta_2 \text{ y } R_m \neq R_F \tag{6}$$

$$x_2^{*} = \frac{-\beta_1}{\beta_2 - \beta_1} + \frac{(R_d - R_F)}{(R_m - R_F)}\,\frac{1}{(\beta_2 - \beta_1)} \tag{7}$$

The portfolios created by following an investment strategy, be it for asset 1 or 2 (each of these having β_1 and β_2), that generates a return equal to $R_F + \beta_k\,(E(R_m) - R_F)$ and is financed with personal equity and short sales at a cost equal to $R_F + \beta_i(E(R_m) - R_F)$, are as much in the efficient frontier as the Capital Market Line. Moreover, such portfolios have desired return of R_d and minimum risk.

In order to know that solutions (6) and (7) are in the efficient frontier, we must show that the portfolio's risk is at a minimum and unique.

In order to show that this portfolio is on the Capital Market Line (CML), we must show that it offers the same desired return, R_d, and risk as a Portfolio in the Efficient Frontier.

Proof

It is known that the CML is: $R_j = R_F + \frac{\sigma_j}{\sigma_m}\left(E(R_m) - R_F\right)$

σ_j = Standard Deviation of alternative portfolio made up of the proportions x'$_1$ and x'$_2$ or:

$$\sigma_j^2 = (x_1')^2(\beta_1\sigma_m)^2 + (1-x_1')^2(\beta_2\sigma_m)^2 + 2x_1'(1-x_1')\beta_1\beta_2\sigma_m^2$$

By replacing the values of x'$_1$ and x'$_2$, and performing algebraic operations, we arrive at:

$$\sigma_j = \sigma_m\left(\frac{R_d - R_F}{R_m - R_F}\right) \qquad \textit{and}$$

by replacing in the CML, we arrive at:

$$R_j = R_F + \frac{\sigma_m}{\sigma_m}\frac{(R_d - R_F)}{(R_m - R_F)}(R_m - R_F)$$

In other words $R_j = R_d$, which in turn is equal to the return obtained in the Portfolio in the Efficient Frontier. Therefore, by using the Efficient Frontier or the CML, we finance the alternative portfolio by investing and financing with debt or short sales, and if CAPM is verified, then it has the same return and risk. The Efficient Frontier and the CML, then, lead to the same conclusion.

We can also see that, in x'$_1$ and x'$_2$, if $R_d = R_F$, we arrives at the same conclusion as Proposition I, in other words, the desired return of the portfolio is the riskless rate. Now, if $R_d = R_m$, then we arrive at Proposition II, the desired return of this formulation being the general case and Propositions I and II being specific cases. On the other hand, from the general expressions of x_1' and x_2' we saw that the definitions of the riskless Portfolio with $\beta = 0$ is a very specific case, as is the Market Portfolio with $\beta = 1$. Both portfolios can be created with assets having $\beta \neq 0$ and $\beta \neq 1$ and will achieve the same results, as can be seen in this paper.

5. Investing and Financing the New Portfolio. General case

Proposition II has already shown that the new portfolio can have an asset, be it financed with debt or short sales plus personal equity, based on the Investment = Financing identity, which breaks down into:

Investment (in Asset 1 or 2) = Debt (or short sale) + Personal equity.

If we have $C of personal equity, we may see the following three situations:

a) $0 < x_1, < 1$ and $0 < x_2, < 1$

In this case:

Investment (in $): $x_1C + x_2C$

Financing (in $): C

Or: x_1' C + x_2C = C

b) $x_1, > 1$ and $x_2, < 0$

In this case:

Investment (in $): x_1C

Financing (in $): $x_2C + C$

Or: x_1'C = x2'C + C

c) $x_1, < 0$ and $x_2, > 0$

In this case:

Investment (in $): x_2C

Financing (in $): x_1C+C

Or: $x_2C = x_1C + C$

The investments' return and financing cost is as follows:

The return of investing in asset i is: $R_F + \beta_i(E(R_m) - R_F)$ i = 1 or 2

The financing cost for asset k is: $R_F + \beta_k(E(R_m) - R_F)$ k = 1 or 2

Personal equity return: R_d

The three situations mentioned above are directly derived from the relationship $x_1+x_2=1$, since if $x_1 > 1$ and $x_2 < 0 \Rightarrow x_1 = x_2+1$; or $x_1 < 0$ and $x_2 > 1 \Rightarrow x_2 = x_1 + 1$

We can see that the alternative portfolio can be widened for i = 1,... k; so long as the Lagrange function is set out as follows:

$$L = \sigma_m^2 \sum_{i=1}^{k} x_i^2 \beta_i^2 + 2\sigma_m^2 \sum_{i=1}^{k}\sum_{j=1}^{k} x_i x_j \beta_i \beta_j + \lambda\left[R_a - \sum_{i=1}^{k} x_i R_i\right]$$

We can clarify the above situations with the following example: there are two risky assets in which we may invest or make short sales.

Appendix Nº 1

Relationship between σ_{ij} with β_{im} and β_{jm}

For the portfolio's function of risk, we must know $\sigma_{ij} = f(\beta_{im}, \beta_{jm},)$, where:

σ_{ij} = Covariance between the returns of asset i and j.

β_{km} = Coefficient β between the returns of asset k and the market portfolio m, k = 1,...j

Proof

Assuming two assets, i and j, whose returns R_k have the following relationships:

$$R_i = \alpha_i + \beta_{im} R_m \quad (1)$$

$$R_j = \alpha_j + \beta_{jm} R_m \quad (2)$$

Where R_m = Return of a market portfolio.

For (1) and (2) we arrive at:

$$R_m = \frac{R_i - \alpha_i}{\beta_{im}} \quad \text{and} \quad R_m = \frac{R_j - \alpha_j}{\beta_{jm}}$$

By balancing, we arrive at:

$$\frac{R_i - \alpha_i}{\beta_{im}} = \frac{R_j - \alpha_j}{\beta_{jm}}$$

By clearing R_i we arrive at:

$$R_i = \frac{\beta_{jm}\alpha_i - \beta_{im}\alpha_j}{\beta_{jm}} + \left(\frac{\beta_{im}}{\beta_{jm}}\right) R_j \quad (3)$$

Likewise, it is known that a relationship can be established of the following type:

$$R_i = \alpha_o + \beta' R_j$$

Where:

$$\beta' = \frac{\sigma_{ij}}{\sigma_j^2} \quad (4)$$

Of (3) it is known that:

$$\beta' = \frac{\beta_{im}}{\beta_{jm}}$$

Or:

$$\frac{\sigma_{ij}}{\sigma_j^2} = \frac{\beta_{im}}{\beta_{jm}}$$

By clearing:

$$\sigma_{ij} = \frac{\beta_{im}}{\beta_{jm}} \sigma_j^2$$

Because asset j is in equilibrium, then:

$$\sigma_j^2 = \left(\beta_{jm}\sigma_m\right)^2$$

From which we can see that:

$$\sigma_{ij} = \beta_{im}\beta_{jm}\sigma_m^2$$

Appendix N° 2

Calculation of proportions x1 and x2 that minimize the risk of a portfolio with two risky assets

We know that:

$$\sigma_c^2 = x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2\sigma_{12}$$

Where σ_1^2 *and* σ_2^2*, the variance of the returns of assets 1 and 2, respectively,* σ_{12} *= Covariance of the returns of assets 1 and 2*

We know that:

$x_2 = x_1 - 1$ (from $x_1 + x_2 = 1$)

$\sigma^2_1 = (\beta_1\sigma_m)^2$ (When in equilibrium, according to CAPM)

$\sigma^2_2 = (\beta_2\sigma_m)^2$ (When in equilibrium, according to CAPM)

$\sigma_{12} = \beta_1\beta_2\sigma^2_m$ (from Appendix 1)

Assets 1 and 2 are two risky assets.

So, the function to be optimized is:

$$\sigma_c^2 = x_1^2\left(\beta_1\sigma_m\right)^2 + (1-x_1)^2\left(\beta_2\sigma_m\right)^2 + 2x_1\left(1-x_1\right)\beta_1\beta_2\sigma_m^2$$

By using a minimization process, we arrive at:

$$\frac{\partial\sigma_c^2}{\partial x_1} = 2x_1(\beta_1\sigma_m)^2 - 2(1-x_1)(\beta_2\sigma_m)^2 + 2(1-x_1)\beta_1\beta_2\sigma_m^2 - 2x_1\beta_1\beta_2\sigma_m^2 = 0$$

By clearing for x_1, the optimum value (x^*_1) for x_1 is:

$$2\sigma_m^2 x_1(\beta_1^2 - 2\beta_1\beta_2 + \beta_2^2) = 2\sigma_m^2(\beta_2^2 - \beta_1\beta_2)$$

Since $x_1 + x_2 = 1$, then the optimum value for x_2 is:

$$x_2^* = \frac{-\beta_1}{\beta_2 - \beta_1} \quad \text{with } \beta_2 \neq \beta_1$$

We have shown that:

$$\frac{\partial_2\sigma_c^2}{\partial x_1^2} = 2\sigma_m^2(\beta_1 - \beta_2)^2 > 0$$

So σ^2_c in (x^*_1,x^*_2) is at a minimum.

Appendix N° 3

Calculation of an alternative market portfolio, investment proportions, and financing

We know that:

$$L = x_1^2(\beta_1\sigma_m)^2 + (1-x_1)^2(\beta_2\sigma_m)^2 + 2x(1-x)\beta_1\beta_2\sigma_m^2 + \lambda\left[R_m - x_1R_1 - (1-x_1)R_2\right]$$

Where: R_m = market portfolio return, with any Beta

λ = Lagrange multiplier.

The assumptions of Appendix No. 2 are maintained.

Our idea is to try to find the x^*_1 and x^*_2 proportions that conform a portfolio having minimum risk and a return equivalent to the return of a market portfolio R_m. To do so, we calculate the Lagrange multipliers, for which the First Order conditions are the following:

$$\partial L/\partial x_1 = 2x_1\sigma^2_m(\beta_2-\beta_1)^2 + \lambda[R_2 - R_1] - 2\sigma^2_m\beta_2(\beta_2-\beta_1) = 0$$

$$\partial L/\partial \lambda = x_1(R_2-R_1) \quad - \quad (R_2-R_m) = 0$$

To solve this system of equations, we must use the following matrix procedure:

$$\partial L/\partial \lambda = x_1(R_2-R_1) \quad - \quad (R_2-R_m) = 0$$

x_1	λ	Constant
$2\sigma^2_m(\beta_2-\beta_1)^2$ R_2-R_1	R_2-R_1 0	$2\sigma^2_m\beta_2(\beta_2-\beta_1)$ R_2-R_m
1	$\frac{R_2-R_1}{2\sigma^2_m(\beta_2-\beta_1)^2}$	$\frac{\beta_2}{\beta_2-\beta_1}$
0	$\frac{-(R_2-R_1)^2}{2\sigma^2_m(\beta_2-\beta_1)^2}$	$\frac{-\beta_2(R_2-R_1)}{\beta_2-\beta_1}+(R_2-R_m)$

Based on this last matrix, we can establish the following relationship between x_1 and λ:

$$x_1 + \lambda\left[\frac{R_2-R_1}{2\sigma^2_m(\beta_2-\beta_1)^2}\right] = \frac{\beta_2}{\beta_2-\beta_1} \tag{1}$$

$$\lambda\left[\frac{-(R_2-R_1)^2}{2\sigma^2_m(\beta_2-\beta_1)^2}\right] = -\frac{\beta_2(R_2-R_1)}{\beta_2-\beta_1} + (R_2-R_m) \tag{2}$$

by clearing λ from (2) and substituting it in (1), we arrive at the following x_1 value:

$$x^*_1 = \frac{R_2-R_m}{R_2-R_1} \quad with\ R_2 \neq R_1$$

By replacing the returns of assets 1 and 2 with those obtained according to CAPM, we arrive at:

$$x^*_1 = \frac{R_F + \beta_2[R_m - R_F] - R_m}{R_F + \beta_2[R_m-R_F] - R_F - \beta_1[R_m - R_F]}$$

By reducing, we are left with the following:

$$x^*_1 = \frac{\beta_2 - 1}{\beta_2-\beta_1} \quad with\ \ \beta_2 \neq \beta_1$$

Since $x^*_1 + x^*_2 = 1$, then

$$x^*_2 = \frac{1-\beta_1}{\beta_2-\beta_1} \quad with\ \beta_2 \neq \beta_1$$

The second order conditions imply that:

$$\partial_2 L(x_1) = \frac{\partial^2 F}{\partial x^2_1} = 2\sigma^2_M(\beta_2-\beta_1)^2 > 0, therefore, it\ is\ a\ conditioned\ minimum.$$

Appendix N° 4

We took this formulation directly from Appendix No. 3, replacing the x^*_1 value with the R_m value for R_d in x^*_1, or:

$$x'_1 = \frac{R_F + \beta_2[R_m - R_F] - R_d}{\beta_2[R_m-R_F] - \beta_1[R_m-R_F]}$$

By arranging, we arrive at:

$$x'_1 = \frac{\beta_2}{\beta_2 - \beta_1} - \frac{(R_d - R_F)}{(R_m - R_F)} \frac{1}{(\beta_2 \beta_1)} \quad with\ \beta_1 \neq \beta_2\ y\ R_m \neq R_F$$

Since x’$_1$ + x’$_2$ = 1, then x’$_2$ is:

$$x'_2 = \frac{-\beta_1}{\beta_2 - \beta_1} + \frac{(R_d - R_F)}{(R_m - R_F)} \cdot \frac{1}{(\beta_2 - \beta_1)}$$

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